

## CYCLE RELATIONS ON JACOBIAN VARIETIES

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ABSTRACT. By using the Grothendieck-Riemann-Roch theorem we derive cycle relations modulo algebraic equivalence in the Jacobian of a curve. The relations generalize the relations found by Colombo and van Geemen and are analogous to but simpler than the relations recently found by Herbaut. In an appendix due to Zagier it is shown that these sets of relations are equivalent.

## 1. INTRODUCTION

Beauville showed in [1] that the Chow ring with rational coefficients of an abelian variety possesses a double grading  $CH_{\mathbb{Q}}(X) = \oplus CH_{(j)}^i(X)$  where  $i$  refers to the codimension and  $j$  refers to the action of the integers:  $CH_{(j)}^i(X) = \{x \in CH^i(X) : k^*(x) = k^{2i-j}x\}$ . The quotient  $A(X)$  of the Chow ring modulo algebraic equivalence inherits this double grading  $A(X) = \oplus A_{(j)}^i(X)$  and carries two multiplication laws, the intersection product  $x \cdot y$  and the Pontryagin product  $x * y$ .

If  $X = \text{Jac}(C)$  is the Jacobian of a curve  $C$  of genus  $g$  then we can decompose the class  $[C]$  of the image of the Abel-Jacobi map of  $C$  as  $[C] = \sum_{j=0}^{g-1} C_{(j)}$  with  $C_{(j)} \in A_{(j)}^{g-1}(\text{Jac}(C))$ . Colombo and van Geemen proved (cf. [3]) that for a curve  $C$  with a map of degree  $d$  to  $\mathbf{P}^1$  the component  $C_{(j)}$  vanishes for  $j \geq d-1$ . In [5] Herbaut extended this result and found cycle relations for curves having a  $g_d^r$ , i.e., a linear system of degree  $d$  and projective dimension  $r$ , with  $r \geq 2$ .

It is the purpose of this note to show that one can use the Grothendieck-Riemann-Roch theorem to derive in an easy way the Colombo-van Geemen result as well as simple relations of higher degree.

Let  $C$  be a smooth projective curve of genus  $g$  over an algebraically closed field  $K$ .

**Theorem 1.1.** *If  $C$  has a base point free  $g_d^r$  then*

$$\sum_{a_1 + \dots + a_r = N} (a_1 + 1)! \cdots (a_r + 1)! C_{(a_1)} * \cdots * C_{(a_r)} = 0,$$

for every  $N \geq d - 2r + 1$ .

For  $N = d - 2r + 1$  this relation coincides with Herbaut's relation (cf., [5], Thm 1 and Thm 8). For higher values of  $N$  they are in general different, but in an appendix we present a proof by Don Zagier that shows that these sets of relations are equivalent.

## 2. PRELIMINARIES

Let  $C$  be a smooth projective curve of genus  $g$  over an algebraically closed field  $K$ . We suppose that the curve  $C$  has a base-point free linear system  $g_d^r$  of degree  $d$

and projective dimension  $r$ . This defines a morphism  $\gamma : C \rightarrow \mathbf{P}^r$ . Let  $J = \text{Jac}(C)$  be the Jacobian of  $C$ .

We consider the incidence variety  $Y \subset C \times \hat{\mathbf{P}}^r$  defined by

$$Y = \{(p, \eta) \in C \times \hat{\mathbf{P}}^r : \gamma(p) \in \eta\},$$

where  $\hat{\mathbf{P}}^r$  is the dual projective space of  $\mathbf{P}^r$ . It has dimension  $r$  and possesses the two projections  $\tilde{\phi}$  and  $\tilde{\alpha}$  onto  $C$  and  $\hat{\mathbf{P}}^r$ . Note that  $\tilde{\alpha}$  is finite of degree  $d$  and  $\tilde{\phi}$  is a  $\mathbf{P}^{r-1}$ -fibration. We shall write  $\mathbf{P}^r$  for  $\hat{\mathbf{P}}^r$ .

We have the following diagram of morphisms

$$\begin{array}{ccccccc} \mathbf{P}^r & \xleftarrow{v} & \mathbf{P}^r \times J & \xleftarrow{\alpha} & Y \times J & \xrightarrow{\pi} & Y & \xrightarrow{\tilde{\alpha}} & \mathbf{P}^r \\ & & \downarrow p & & \downarrow \phi & & \downarrow \tilde{\phi} & & \\ & & J & \xleftarrow{q} & C \times J & \xrightarrow{\tilde{\pi}} & C & & \end{array}$$

where the morphisms  $v, p, q, \tilde{\pi}$  and  $\pi$  are projections and  $\alpha = \tilde{\alpha} \times \text{id}_J$  and  $\phi = \tilde{\phi} \times \text{id}_J$ .

Let  $P$  be the Poincaré bundle on  $C \times J$  and set  $L := \phi^* P$ , a line bundle on  $Y \times J$ . Put  $\ell := c_1(L)$  and  $\Pi := c_1(P)$ .

The Chow ring of  $\mathbf{P}^r \times J$  is generated over  $CH^*(J)$  by the class  $\xi = v^* h$  with  $h$  a hyperplane in  $\mathbf{P}^r$  with  $\xi^{r+1} = 0$ . For a class  $\beta \in CH^*(\mathbf{P}^r \times J)$  we have the relation (cf. [4], (Thm. 3.3, p. 64))

$$\beta = \sum_{i=0}^r \beta_i \xi^{r-i} \quad \text{with} \quad \beta_i = p_*(\beta \cdot \xi^i), \quad (1)$$

where by abuse of notation we write here and hereafter  $\beta_i$  for  $p^*(\beta_i)$ .

We let  $x = \alpha^*(\xi)$  be the pull back of  $\xi$ . We let  $\rho = \pi^* \tilde{\phi}^*(\text{point})$  be the pull back class of a point on  $C$ . We work in the Chow ring up to algebraic equivalence. There we have the relations

$$x^r = d \pi^*(\text{point}), \quad x^{r+1} = 0, \quad \rho^2 = 0, \quad x^{r-1} \rho = \pi^*(\text{point}).$$

Recall the Fourier transform  $F : A(X) \rightarrow A(X)$  for a principally polarized abelian variety  $(X, \theta)$  of dimension  $g$ , cf. [1, 2]. It has the properties i)  $F \circ F = (-1)^g (-1)^*$ , ii)  $F(x * y) = F(x) \cdot F(y)$  and  $F(x \cdot y) = (-1)^g F(x) * F(y)$ , iii)  $F(A_{(j)}^i(X)) = A_{(j)}^{g-i+j}(X)$ .

We have the relation  $q_*(e^\Pi) = F[C]$ , cf. [2], Section 2. Comparing terms gives that  $F[C_{(j)}] = (1/(j+2)!) q_*(\Pi^{j+2})$  for  $j = 0, \dots, g-1$ . Note, also, that  $q_* 1 = q_* \Pi = 0$ . More generally, extending scalars to  $\mathbb{Q}$  we have the relation

$$q_*(e^{k\Pi}) = k^{2g} F[(k^{-1})^* C] \quad \text{for } k \in \mathbb{Z}_{\geq 1}.$$

In fact, writing  $[C] = \sum_{j=0}^{g-1} C_{(j)}$  we have  $(k^{-1})^*[C] = \sum_j k^{j+2-2g} C_{(j)}$ , hence

$$k^{2g} F[(k^{-1})^* C] = F\left[\sum_j k^{j+2} C_{(j)}\right] = \sum_j k^{j+2} q_*(\Pi^{j+2}/(j+2)!) = q_*(e^{k\Pi}).$$

### 3. THE PROOF

We shall prove that if  $C$  has a base point free  $g_d^r$  then

$$\sum_{a_1 + \dots + a_r = N} (a_1 + 1)! \cdots (a_r + 1)! F[C_{(a_1)}] \cdots F[C_{(a_r)}] = 0, \quad (2)$$

for every  $N \geq d - 2r + 1$ .

We are going to apply the Grothendieck-Riemann-Roch theorem to the morphism  $\alpha$  and the line bundle  $L$ . For  $k \geq 1$  we put  $V_k := \alpha_*(L^{\otimes k})$ . Since  $\alpha$  is a finite morphism of degree  $d$  this is a vector bundle of rank  $d$  and we get

$$\mathrm{ch}(V_k) = \mathrm{ch}(\alpha_! L^{\otimes k}) = \alpha_*(e^{k\ell} \mathrm{td}_\alpha),$$

with  $\mathrm{td}_\alpha$  the Todd class of the morphism  $\alpha$ .

The Todd class  $\mathrm{td}_\alpha$  is algebraically equivalent to a class of the form  $A(x) + B(x)\rho$ . Here  $A = \sum_{j=0}^{r-1} a_j x^j$  and  $B = \sum_{j=0}^{r-1} b_j x^j$  are polynomials in  $x$  and  $a_0 = 1$ . In fact,  $\mathrm{td}_\alpha$  is the pull back under  $\pi$  of  $\mathrm{td}_{\tilde{\alpha}}$ , an element of  $A(Y)$ . The ring  $A(Y)$  is generated as an  $A(C)$ -module by  $1, x_1, \dots, x_1^{r-1}$  with  $x_1 = \tilde{\alpha}^*(h)$  and  $A(C)$  is generated by 1 and the class of a point.

**Proposition 3.1.** *For  $k \in \mathbb{Z}_{\geq 1}$  we have in  $A(\mathbf{P}^r \times J)$  the relation*

$$\mathrm{ch}(V_k) = dA(\xi) + \xi B(\xi) + k^{2g} F[(k^{-1})^* C] \xi A(\xi).$$

*In particular, all  $\mathrm{ch}_j(V_k)$  are divisible by  $\xi$  for  $j \geq 1$ .*

Before we give the proof of Proposition 3.1 we state a corollary and a lemma.

Proposition 3.1 gives an expression of the Chern characters of the bundles  $V_k$ . We can express the Chern classes of the bundles  $V_k$  of rank  $d$  in terms of the Chern characters by using the well-known formula (cf. [6], ch. I (2.10'))

$$1 + c_1(V_k)t + \dots + c_d(V_k)t^d = \exp\left(\sum_{j \geq 1} (-1)^{j-1} (j-1)! \mathrm{ch}_j(V_k) t^j\right). \quad (3)$$

Formula (3) combined with Proposition 3.1 will give us the vanishing relations we are asking for. For example, applying these formulas for  $r = 1$  and  $k = 1$  immediately gives us the Theorem of Colombo-van Geemen [3] as we now show.

**Corollary 3.2.** *If  $C$  has a  $g_d^1$  then  $C_{(j)} = 0$  for  $j \geq d - 1$ .*

*Proof.* Put  $V = V_1$ . We see  $\mathrm{ch}(V) = d + n\xi + F[C]\xi$  for some  $n$  (actually  $n = 1 - d - g$ ). Since  $\mathrm{ch}_j(V)$  is divisible by  $\xi$  for  $j \geq 1$  and  $\xi^2 = 0$ , formula (3) becomes in this case:  $1 + c_1(V)t + \dots + c_d(V)t^d = 1 + \mathrm{ch}_1(V)t - \mathrm{ch}_2(V)t^2 + \dots + (-1)^{j-1} (j-1)! \mathrm{ch}_j(V) t^j + \dots$ . Therefore  $\mathrm{ch}_j(V)$  vanishes for  $j > d$ . Hence  $F[C]$  has no terms of codimension  $\geq d$ . Since  $F[C_{(j)}]$  is of codimension  $j + 1$ , it follows that  $C_{(j)} = 0$  for all  $j \geq d - 1$ .  $\square$

**Lemma 3.3.** *In  $A(\mathbf{P}^r \times J)$  the following relations hold for  $\nu \geq 0$  :*

$$\alpha_*(\ell^\mu \cdot x^\nu) = \begin{cases} q_*(\Pi^\mu) \xi^{\nu+1} & \mu > 0, \\ d \xi^\nu & \mu = 0 \end{cases} \quad \text{and} \quad \alpha_*(\ell^\mu \cdot x^\nu \cdot \rho) = \begin{cases} 0 & \mu > 0, \\ \xi^{\nu+1} & \mu = 0. \end{cases}$$

*Proof.* For the first relation: By (1) the coefficient of  $\xi^{r-j}$  is given by

$$\begin{aligned} p_*(\alpha_*(\ell^\mu x^\nu) \xi^j) &= p_*(\alpha_*(\ell^\mu x^\nu \alpha^*(\xi)^j)) = p_*(\alpha_*(\phi^*(\Pi)^\mu x^{\nu+j})) \\ &= q_*(\phi_*(\phi^*(\Pi)^\mu x^{\nu+j})) = q_*(\Pi^\mu \phi_*(x^{\nu+j})). \end{aligned}$$

If  $\nu + j = r$  then  $x^r$  is algebraically equivalent to  $d$  times point  $\times J$  and since  $\Pi$  is algebraically equivalent to 0 on point  $\times J$  we get that any term with  $\mu > 0$  and  $\nu + j = r$  contributes 0. The term with  $\nu + j = r - 1$  contributes  $q_*(\Pi^\mu \phi_*(x^{r-1})) = q_*(\Pi^\mu)$  since  $\phi_*(x^{r-1}) = 1_{C \times J}$ . If  $\nu + j < r - 1$  or  $\nu + j > r$  then we get  $q_*(\Pi^\mu \phi_*(x^{\nu+j})) = q_*(\Pi^\mu 0) = 0$ . Finally, if  $\mu = 0$  we use that  $x = \alpha^* \xi$ , hence  $\alpha_*(x^\nu) = d \xi^\nu$ .

For the second relation: Observe that  $\ell^\mu \cdot \rho = 0$ , if  $\mu \geq 1$ . Indeed,  $\ell = \phi^* \Pi$ ,  $\rho = \phi^*(\text{point} \times J)$  and  $\Pi \cdot (\text{point} \times J) = 0$ . When  $\mu = 0$  we have  $\alpha_*(x^\nu \rho) = \xi^\nu \alpha_* \rho = \xi^{\nu+1}$ .

□

We now give the proof of Proposition 3.1.

*Proof.* We have  $\alpha_*(\ell^\mu x^\nu \rho) = 0$  for all  $\mu \geq 1, \nu \geq 0$ . So in the contributions  $\alpha_*(e^{k\ell} \text{td}_\alpha)$  we get contributions of the form  $\alpha_*(e^{k\ell} A(x))$  and  $\alpha_*(\rho B(x))$  only. We have

$$\begin{aligned} \alpha_*(e^{k\ell} A(x)) &= \alpha_*(A(x)) + \sum_{\mu \geq 1} \frac{k^\mu}{\mu!} \alpha_*(\ell^\mu A(x)) = dA(\xi) + \sum_{\mu \geq 1} \frac{k^\mu}{\mu!} q_* \Pi^\mu \xi A(\xi) \\ &= dA(\xi) + q_*(e^{k\Pi}) \xi A(\xi) = dA(\xi) + k^{2g} F[(k^{-1})^* C] \xi A(\xi). \end{aligned}$$

On the other hand,  $\alpha_*(\rho B(x)) = \xi B(\xi)$ .

□

By using the relation

$$k^{2g} F[(k^{-1})^* C] = q_*(e^{k\Pi}) = \sum_{\mu \geq 0} \frac{k^{\mu+2}}{\mu+2!} q_* \Pi^{\mu+2} = \sum_{\mu \geq 0} k^{\mu+2} F[C_{(\mu)}],$$

we get the following corollary of Proposition 3.1.

**Corollary 3.4.** *We put  $a_j = b_j = 0$  for every  $j \geq r$ . With  $A = \sum_{j=0}^{r-1} a_j x^j$  where  $a_0 = 1$  and  $B = \sum_{j=0}^{r-1} b_j x^j$  we have for  $j \geq 1, k \geq 1$  that*

$$\text{ch}_j(V_k) = (d a_j + b_{j-1}) \xi^j + \sum_{m=1}^{j-1} a_{m-1} k^{j-m+1} F[C_{(j-m-1)}] \xi^m.$$

With  $j \geq 1$  we put

$$\text{ch}_j(V_k) = A_1(j) \xi^1 + \cdots + A_r(j) \xi^r,$$

where  $A_m(j)$  is of codimension  $j - m$ . Please note that  $\text{ch}_j(V_k)$  is divisible by  $\xi$  for  $j \geq 1$  by Prop. 3.1.

**Remark 3.5.** The coefficient  $A_m(j)$  depends on  $k$ , but for simplicity of notation we do not involve the index  $k$  in the notation.

Then, for every  $j \geq 1$  and  $1 \leq m \leq r$ , we have

$$A_m(j) = \begin{cases} d a_j + b_{j-1} & m = j, \\ a_{m-1} k^{j-m+1} F[C_{(j-m-1)}] & m < j, \\ 0 & m > j. \end{cases}$$

Since  $\text{rank}(V_k) = d$  the coefficient of  $t^{M+1}$  of the right hand side of (3) must be zero for every  $M \geq d$ . Let us write

$$F(t) = \sum_{j \geq 1} (-1)^{j-1} (j-1)! \text{ch}_j(V_k) t^j.$$

Note that  $F(t)^i = 0$  for every  $i \geq r+1$  because  $\text{ch}_j(V_k)$  is divisible by  $\xi$  for  $j \geq 1$ . Therefore the right hand side of (3) is equal to  $\sum_{i=0}^r (1/i!) F(t)^i$ . The coefficient of  $t^{M+1}$  in the polynomial  $F(t)^i$ , for  $i \geq 1$ , is equal to

$$(-1)^{M+1-i} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} (\alpha_1 - 1)! \cdots (\alpha_i - 1)! \text{ch}_{\alpha_1}(V_k) \cdots \text{ch}_{\alpha_i}(V_k).$$

We denote the expression  $(\alpha_1 - 1)! \cdots (\alpha_i - 1)!$  by  $\alpha\{1, i\}$ . Then we have that

$$\sum_{i=1}^r \frac{(-1)^i}{i!} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} \operatorname{ch}_{\alpha_1}(V_k) \cdots \operatorname{ch}_{\alpha_i}(V_k) = 0$$

for  $M \geq d$  and so,

$$\sum_{i=1}^r \frac{(-1)^i}{i!} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} \left[ \sum_{m_1=1}^r A_{m_1}(\alpha_1) \xi^{m_1} \right] \cdots \left[ \sum_{m_i=1}^r A_{m_i}(\alpha_i) \xi^{m_i} \right] = 0.$$

Now the LHS of this is easily seen to be equal to

$$\sum_{m=1}^r \sum_{i=1}^m \frac{(-1)^i}{i!} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} \sum_{m_1 + \cdots + m_i = m} [A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i)] \xi^m.$$

We therefore have for every  $m = 1, \dots, r$  and  $M \geq d$  that

$$\sum_{i=1}^m \frac{(-1)^i}{i!} \sum_{m_1 + \cdots + m_i = m} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i) = 0.$$

With  $M \geq d$  the case  $m = r$  gives the relation

$$\sum_{i=1}^r \frac{(-1)^i}{i!} \sum_{m_1 + \cdots + m_i = r} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i) = 0.$$

If we write

$$B_M(i) = \sum_{m_1 + \cdots + m_i = r} \sum_{\alpha_1 + \cdots + \alpha_i = M+1} \alpha\{1, i\} A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i)$$

then this relation becomes  $\sum_{i=1}^r ((-1)^i / i!) B_M(i) = 0$  for every  $M \geq d$ . We analyze the dependence on  $k$ .

**Proposition 3.6.** *We write  $\sum_{i=1}^r \frac{(-1)^i}{i!} B_M(i) = \sum_s \Gamma_s k^s$  as a polynomial in  $k$ . With  $M \geq d$  we have that  $\Gamma_s = 0$  for  $s > M + 1$  and*

$$\Gamma_{M+1} = \frac{(-1)^r}{r!} \sum_{\alpha_1 + \cdots + \alpha_r = M-2r+1} (\alpha_1 + 1)! \cdots (\alpha_r + 1)! F[C_{a_1}] \cdots F[C_{a_r}].$$

*Proof.* If  $A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i)$  contains a factor with  $m_j > \alpha_j$  then it vanishes. Otherwise, since  $A_{m_j}(\alpha_j) = a_{m_j-1} k^{\alpha_j - m_j + 1} F[C_{\alpha_j - m_j - 1}]$ , except when  $m_j = \alpha_j$ , in which case  $A_{\alpha_j}(\alpha_j) = d a_{\alpha_j} + b_{\alpha_j-1}$ , the power of  $k$  contained in  $A_{m_1}(\alpha_1) \cdots A_{m_i}(\alpha_i)$  is equal to  $(\alpha_1 + \cdots + \alpha_i) - (m_1 + \cdots + m_i) + \nu = M + 1 - r + \nu$ , where  $\nu$  is given by  $\nu = \#\{m_j \neq \alpha_j, j = 1, \dots, i\} \leq i$ . Now if  $i < r$  the above number is  $< M + 1$ . On the other hand, if  $i = r$  then  $m_1 = \cdots = m_r = 1$  (since  $m_i \geq 1$ ) and therefore

$$B_M(r) = \sum_{\alpha_1 + \cdots + \alpha_r = M+1} \alpha\{1, r\} A_1(\alpha_1) \cdots A_1(\alpha_r).$$

Again, if a term of the above sum contains a factor  $A_1(1)$ , then the power of  $k$  contained in this term is  $< M + 1$ . On the other hand, the sum of the terms with no factor of the form  $A_1(1)$  is

$$B'_M(r) = \sum_{a_\mu \geq 2, \alpha_1 + \cdots + \alpha_r = M+1} \alpha\{1, r\} A_1(\alpha_1) \cdots A_1(\alpha_r).$$

Since  $\alpha_\mu \geq 2$  we have that  $A_1(\alpha_\mu) = k^{\alpha_\mu} F[C_{\alpha_\mu-2}]$ . Therefore

$$\begin{aligned} B'_M(r) &= k^{M+1} \sum_{\alpha_\mu \geq 2, \alpha_1 + \dots + \alpha_r = M+1} \alpha\{1, r\} F[C_{\alpha_1-2}] \cdots F[C_{\alpha_r-2}] \\ &= k^{M+1} \sum_{\alpha_\mu \geq 0, \alpha_1 + \dots + \alpha_r = M-2r+1} (\alpha_1 + 1)! \cdots (\alpha_r + 1)! F[C_{\alpha_1}] \cdots F[C_{\alpha_r}]. \end{aligned}$$

□

We now prove Theorem 1.1

*Proof.* Since we are working with  $\mathbb{Q}$ -coefficients, we conclude that the coefficient of  $k^{M+1}$  (which is the maximum degree of  $k$  involved) must be zero for  $M \geq d$ , which is relation (2) with  $N = M - 2r + 1 \geq d - 2r + 1$ . By applying the Fourier transform Theorem 1.1 follows. □

#### 4. APPENDIX PROVIDED BY DON ZAGIER

**4.1. A combinatorial identity.** We set

$$B_d(a_1, \dots, a_r) = \sum_{i_1, \dots, i_r \geq 1} (-1)^{d-i_1-\dots-i_r} \binom{d}{i_1 + \dots + i_r} i_1^{a_1} \cdots i_r^{a_r}.$$

Using  $\sum_{d=0}^{\infty} (-1)^{d-i} \binom{d}{i} u^d = \frac{u^i}{(1+u)^{i+1}}$  we find that  $\sum_{d=0}^{\infty} B_d(a_1, \dots, a_r) u^d$  equals

$$\sum_{i_1, \dots, i_r \geq 1} i_1^{a_1} \cdots i_r^{a_r} \frac{u^{i_1 + \dots + i_r}}{(1+u)^{i_1 + \dots + i_r + 1}} = \frac{1}{1+u} P_{a_1+1}(u) \cdots P_{a_r+1}(u),$$

where  $P_n(u)$  is the power series

$$P_n(u) := \sum_{i=1}^{\infty} i^{n-1} \left( \frac{u}{1+u} \right)^i \in \mathbb{Z}[[u]] \quad (n \geq 1).$$

**Lemma 4.1.** (i)  $P_n(u)$  is a polynomial of degree  $n$ . More precisely, we have

$$P_n(u) = \sum_{m=1}^n (m-1)! \mathfrak{S}_n^{(m)} u^m \text{ where each } \mathfrak{S}_n^{(m)} \text{ is a positive integer.}$$

(ii)  $P_n(-1) = 0$  for  $n > 1$ .

(iii) For each  $n \geq 1$  one has the Laurent series identity

$$\frac{(n-1)!}{[\log(1+x)]^n} = P_n\left(\frac{1}{x}\right) + O(x) \quad \text{in } \mathbb{Q}[x^{-1}, x]. \quad (4)$$

**Example 4.2.** The first five values of  $P_n(u)$  are  $u, u+u^2, u+3u^2+2u^3, u+7u^2+12u^3+6u^4$  and  $u+15u^2+50u^3+60u^4+24u^5$ . We have

$$\frac{4!}{[\log(1+x)]^5} = \frac{24}{x^5} + \frac{60}{x^4} + \frac{50}{x^3} + \frac{15}{x^2} + \frac{1}{x} + 0 - \frac{x}{252} + \frac{x^2}{504} - \frac{19x^3}{30240} - \frac{x^4}{20160} + \frac{53x^5}{147840} - \cdots.$$

*Proof.* The easiest approach (as usual!) is to use generating functions. We have

$$\sum_{n=1}^{\infty} P_n(u) \frac{t^{n-1}}{(n-1)!} = \sum_{i=1}^{\infty} \left( \frac{ue^t}{1+u} \right)^i = \frac{ue^t}{1-u(e^t-1)} \quad (5)$$

and hence

$$\frac{1}{(n-1)!} \mathbf{C}_{u^m}[P_n(u)] = \mathbf{C}_{t^{n-1}}[e^t(e^t-1)^{m-1}] \quad (m, n \geq 1).$$

(Here  $\mathbf{C}_{x^i}[\Phi(x)]$  denotes the coefficient of  $x^i$  in a polynomial, power series or Laurent series  $\Phi(x)$ .) This clearly vanishes for  $n < m$ , showing that  $P_n$  is a polynomial of degree  $\leq n$ , and we also get the explicit formula for  $P_n$  given in (i), with

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!} \mathbf{C}_{t^n}[(e^t - 1)^m] = \mathbf{C}_{x^{n-m}} \left[ \prod_{j=1}^m \frac{1}{1-jx} \right] = \frac{1}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^n$$

being the Stirling number of the second kind (= number of ways of partitioning a set of  $n$  elements into  $m$  non-empty subsets). The right-hand side of (5) reduces to  $-1$  at  $u = -1$ , proving (ii). For (iii), we use the residue theorem and the substitution  $e^t = 1 + x$  to get

$$\frac{1}{(n-1)!} \mathbf{C}_{u^n} [P_n(u)] = \text{Res}_{t=0} \left[ \frac{e^t (e^t - 1)^{m-1} dt}{t^n} \right] = \text{Res}_{x=0} \left[ \frac{x^{m-1} dx}{[\log(1+x)]^n} \right]$$

for  $m, n \geq 1$ .  $\square$

**4.2. The equivalence of the two sets of relations.** Let  $R$  be the  $\mathbb{Q}$ -subalgebra of the Chow algebra generated (with respect to the Pontryagin product) by the  $C_{(j)}$  ( $0 \leq j \leq g-1$ ), bigraded by  $C_{(j)} \in R_{1,j}$ , so that  $R_{ij} \subseteq A_{(j)}^{g-i}$ . We define two polynomials

$$G(t) = \sum_{a=0}^{g-1} (a+1)! C_{(a)} t^{a+2} \in R[t], \quad H(u, t) = \sum_{a=0}^{g-1} P_{a+2}(u) C_{(a)} t^{a+2} \in R[u, t].$$

Then the relations obtained from Thm 1.1 can be written in the form

$$\deg_t [G(t)^s] \leq d - r + s \quad \text{for } 1 \leq s \leq r, \quad (6)$$

while, in view of the formulas in the preceding subsection, Herbaut's (cf. [5], Thm1) can be written in the form

$$\mathbf{C}_{u^{d-r+s}} \left[ \frac{1}{1+u} H(u, t)^s \right] = 0 \quad \text{in } R[t] \quad \text{for } 1 \leq s \leq r. \quad (7)$$

We also introduce the *strengthened Herbaut relations*

$$\deg_u [H(u, t)^s] \leq d - r + s \quad \text{for } 1 \leq s \leq r. \quad (8)$$

Clearly (8) implies (7). (Note that  $H(u, t)^s / (1+u)$  is a polynomial by part (ii) of the lemma.) To see that (6) implies (8), we use equation (4) to obtain

$$G\left(\frac{t}{\log(1+x)}\right) = H\left(\frac{1}{x}, t\right) - \varepsilon(x, t)$$

with  $\varepsilon(x, t) = O(x)$  (in fact  $\varepsilon(x, t) = O(xt^2)$ ) and hence, assuming (6),

$$\begin{aligned} (9) \quad H\left(\frac{1}{x}, t\right)^s i &= \sum_{s'=0}^s \binom{s}{s'} G\left(\frac{t}{\log(1+x)}\right)^{s'} \varepsilon(x, t)^{s-s'} \\ &= \sum_{s'=0}^s O\left(\frac{1}{x^{d-r+s'}}\right) O(x^{s-s'}) = O\left(\frac{1}{x^{d-r+s-2}}\right) \end{aligned}$$

as  $x \rightarrow 0$ , proving (8). To prove that (7) implies (6), we use induction on  $s$ . Assume (7) for some  $s \leq r$  and (6) for all  $s' < s$ . Equation (9) and the inductive assumption give

$$G\left(\frac{t}{\log(1+x)}\right)^s = H\left(\frac{1}{x}, t\right)^s + O\left(\frac{1}{x^{d-r+s-2}}\right)$$

and hence for  $n > d - r + s - 2$

$$\mathbf{C}_{t^n} [G(t)^s] \cdot \mathbf{C}_{x^{-d+r-s}} \left[ \frac{x}{1+x} \frac{1}{[\log(1+x)]^n} \right] = \mathbf{C}_{t^n} \left[ \mathbf{C}_{u^{d-r+s}} \left[ \frac{1}{1+u} H(u, t)^s \right] \right] = 0$$

by (7). Differentiating (4) and using part (i) of the lemma, we find that the second factor on the left equals  $\frac{(d-r+s)!}{(n-1)!} \mathfrak{S}_{n-1}^{(d-r+s)}$ , which is non-zero for  $n > d - r + s$ . Equation (6) follows.

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#### REFERENCES

- [1] A. Beauville: Sur l'anneau de Chow d'une variété abélienne. *Math. Annalen* **273** (1986), 647–651.
- [2] A. Beauville: Algebraic Cycles on Jacobian Varieties. *Compositio Math.* **140** (2004), 683–688.
- [3] E. Colombo, B. van Geemen: Notes on curves in a Jacobian. *Compositio Math.* **88** (1993), 333–353.
- [4] W. Fulton: Intersection Theory. *Ergebnisse der Mathematik*. 3. Folge, **2**, 1998, Berlin, Springer Verlag.
- [5] F. Herbaut: Algebraic cycles on the Jacobian of a curve with a  $g_d^r$ . [math.AG/0606140](#)
- [6] I.G. MacDonald: Symmetric functions and Hall polynomials. 2nd ed. Oxford Science Publications. Oxford: Clarendon Press, (1998).

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